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COMPACT CONVEX SETS AND COMPLEX CONVEXITY

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ABSTRACT

We construct a quasi-Banach space which cannot be given an equivalent plurisubharmonic quasi-norm, but such that it has a quotient by a one-dimensional space which is a Banach space. We then use this example to construct a compact convex set in a quasi-Banach space which cannot be affinely embedded into the space L_0 of all measurable functions.

1. Introduction

A little over ten years ago Roberts ([12], [3]) showed that there exists a compact convex subset K of L_p (where $0 < p < 1$) which has no extreme points; in particular K cannot be affinely embedded into a locally convex space. For other examples and related work see [6], [8], [9], and [15].

The purpose of this paper is to construct a compact convex subset K of a quasi-Banach space which has no extreme-points and cannot be affinely embedded into the space L_0 of all measurable functions. Thus, for example, the still unresolved problem of whether every compact convex set has the fixed point property cannot be reduced to considering L_0.

The construction of the example uses the same basic outline as the original Roberts construction. However in place of needle-points as used by Roberts we introduce analytic needle-points. The set K is an absolutely convex set in a complex quasi-Banach space with the property that every continuous plurisubharmonic function $\Psi: K \to \mathbf{R}$ is constant. In view of the recent interest in plurisubharmonic functions on quasi-Banach spaces (cf. [2], [3]) this also should be of interest.

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In order to make analytic needle-points we construct another example which we believe to be of some interest. We construct a complex quasi-Banach space X which has a one-dimensional subspace L so that X/L is a Banach space, but X cannot be given an equivalent plurisubharmonic quasi-norm. In [7] we defined a quasi-Banach space to be A-convex if it can be given an equivalent plurisubharmonic quasi-norm; A-convexity is equivalent to a form of the Maximum Modulus Principle for vector-valued analytic functions. Thus our example shows that a twisted sum of two Banach spaces need not be A-convex. Similar examples for local convexity in place of A-convexity were constructed independently by Ribe [11], Roberts [14] and the author [5].

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2. Notation

We refer the reader to [9] for the basic properties of quasi-normal spaces. We shall say that a quasi-norm is a p-norm $(0 < p \leqq 1)$ if it satisfies

$$\| x_1 + x_2 \|^p \leqq \| x_1 \|^p + \| x_2 \|^p.$$

A well-known theorem of Aoki and Rolewicz asserts that every quasi-norm is equivalent to a p-norm for some $0 < p \leqq 1$.

Let Δ denote the open unit disc in the complex plane and $\mathbf{T}$ the unit circle. If X is a complex quasi-Banach space a function $f: \Delta \to X$ is called *analytic* if it has a power series expansion

$$f(z) = \sum_{n \geqq 0} x_n z^n, \qquad z \in \Delta,$$

and *harmonic* if

$$f(z) = \sum_{n \geqq 0} x_n z^n + \sum_{n > 0} y_n \bar{z}^n, \qquad z \in \Delta.$$

We denote by $A_0(X)$ the space of functions $f: \bar{\Delta} \to X$ so that f is continuous on $\bar{\Delta}$ and analytic on Δ.

If K is an absolutely convex subset of X a function $\Psi: K \to \mathbf{R}$ is called plurisubharmonic if it is upper-semi-continuous and for every finite-dimensional subspace E of X, Ψ is plurisubharmonic on the relative interior of $E \cap K$. X is called A-convex if it can be given an equivalent purisubharmonic quasi-norm. It is shown in [7] that X is A-convex if for some C and every

$f \in A_0(X)$

$$\| f(0) \| \leqq C \max_{|z|=1} \| f(z) \| .$$

We denote by m normalized Haar measure $d\theta/2\pi$ on $\mathbf{T}$ and let $L_p(\mathbf{T}) = L_p(\mathbf{T}, m)$. The variable $e^{i\theta}$ on $\mathbf{T}$ will also be denoted by w for convenience, so that if $g: \Delta \rightarrow L_p(\mathbf{T})$ is an analytic function then z is used for the variable in Δ and w for the variable in $\mathbf{T}$.

3. Analytic needle-points

In this section we describe a modification of the Roberts technique (cf. [9], [12]) for constructing pathological compact convex sets. Let X be a complex quasi-Banach space. Then $x \in X$ will be called an *analytic needle-point* of X if, given $\varepsilon > 0$, there exists $g \in A_0(X)$ with

(1) $g(0) = x$;

(2) $\| g(z) \| < \varepsilon$, $z \in \mathbf{T}$;

(3) if $y \in \operatorname{co} g(\bar{\Delta})$ there exists α, $0 \leqq \alpha \leqq 1$ with $\| y - \alpha x \| < \varepsilon$.

Note that if X contains a non-zero analytic needle-point then X cannot be A-convex. Before showing that such a situation can occur, we describe the use of such needle-points. For convenience suppose X is p-normed where $0 < p < 1$.

LEMMA 3.1. *Let x be an analytic needle-point of X. Then given any $\varepsilon > 0$ there is a finite set $F = F(x, \varepsilon) \subset X$ and a polynomial $\phi \in A_0(X)$ so that:*

(4) $\phi(\bar{\Delta}) \subset \operatorname{co} F$;

(5) $\phi(0) = x$;

(6) $\| \phi(z) \| < \varepsilon$, $z \in \mathbf{T}$;

(7) *if $y \in \operatorname{co} F$ there exists α, $0 \leqq \alpha \leqq 1$ with $\| y - \alpha x \| < \varepsilon$;*

(8) *if $y \in F$ then $\| y \| < \varepsilon$.*

PROOF. Let $L = \{\alpha x: 0 \leqq \alpha \leqq 1\}$. Pick $g \in A_0(X)$ satisfying (1)–(3) with ε replaced by $3^{-1/p}\varepsilon$. For γ close enough to one, $g(\gamma z)$ fulfills the same properties so we may suppose that

$$g(z) = \sum_{n=0}^{\infty} u_n z^n, \qquad |z| \leqq 1,$$

where $\| u_n \| \leqq M\beta^n$ for some M and $0 < \beta < 1$.

Select N so that

$$M^p \sum_{N+1}^{\infty} \beta^{np} < \tfrac{1}{3}\varepsilon^p,$$

and set $\phi(z) = u_0 + u_1 z + \cdots + u_N z^N$. Clearly (5) holds. If $c_1, \ldots, c_m \geqq 0$ with $\Sigma\, c_j = 1$ and $z_1, \ldots, z_m \in \bar{\Delta}$, then

$$\| \Sigma\, c_j \phi(z_j) - \Sigma\, c_j g(z_j) \|^p \leqq \sum_{N+1}^{\infty} M^p \beta^{np}$$

$$< \tfrac{1}{3}\varepsilon^p.$$

We conclude that (6) holds and that if $y \in \operatorname{co} \phi(\bar{\Delta})$,

$$d(y, L)^p \leqq \tfrac{2}{3}\varepsilon^p$$

where $d(y, L) = \inf_{l \in L} \| y - l \|$.

Now let $E = [u_0, u_1, \ldots, u_N]$ be the at most $(N+1)$-dimensional space generated by ϕ. Now $x = u_0 \in E$. Let $V \subset E$ be the open set of all $v \in E$ so that

$$d(v, \alpha(\mathbf{T}))^p < \frac{1}{3(N+2)}\, \varepsilon^p.$$

Now by Carathéodory's theorem, if $y \in \operatorname{co} V$ there exist $v_1, \ldots, v_{N+2}$ so that $y \in \operatorname{co}\{v_1, \ldots, v_{N+2}\}$. Thus

$$d(y, \operatorname{co} \phi(\mathbf{T})) < \tfrac{1}{3}\varepsilon^p$$

and so

$$d(y, L) < \varepsilon.$$

By a simple compactness argument there is a finite subset F of V with $\phi(\mathbf{T}) \subset \operatorname{int}_E \operatorname{co} F$. Then $\phi(\bar{\Delta}) \subset \overline{\operatorname{co}}\, \phi(\mathbf{T}) = \operatorname{co} \phi(\mathbf{T}) \subset \operatorname{co} F$. If $y \in F$ then

$$\| y \|^p \leqq \tfrac{2}{3}\varepsilon^p + d(y, \phi(\mathbf{T}))^p < \varepsilon^p.$$

REMARK. Conditions (7) and (8) show that x is a needle-point in the sense of Roberts [12].

PROPOSITION 3.2. *Let X be a quasi-Banach space in which every $x \in X$ is an analytic needle-point. Then there is a non-empty compact absoutely convex set $K \subset X$ so that:*

(a) $\operatorname{ext} K = \varnothing$,

(b) *if $h: K \to \mathbf{R}$ is continuous and plurisubharmonic then h is constant.*

PROOF. Fix $\delta_n > 0$ $(n \geqq 1)$ to be any sequence so that $\Sigma\, \delta_n^p < \infty$. Fix any $x_0 \neq 0$ and let $G_0 = \{x_0\}$. We define a sequence of finite sets inductively. If $n \geqq 1$ and G_{n-1} has been selected let us suppose $G_{n-1} = \{y_1, \ldots, y_N\}$ where

$N = |G_{n-1}|$. Let $\varepsilon = N^{-1/p}\delta_n$ and put

$$G_n = \bigcup_{j=1}^{N} F(y_j, \varepsilon)$$

where $F(y_j, \varepsilon)$ is given by Lemma 3.1. We readily verify:

(9) $\mathrm{co}\, G_{n-1} \subset \mathrm{co}\, G_n$;

(10) $d(y, \mathrm{co}\, G_{n-1}) \leqq N\varepsilon^p \leqq \delta_n^p$, $y \in \mathrm{co}\, G_n$.

We shall also need:

(11) If $y \in \mathrm{co}\, G_{n-1}$ *there is a polynomial* $\phi \in A_0(X)$ *with* $\phi(\bar{\Delta}) \subset \mathrm{co}\, G_n$, $\phi(0) = y$ *and* $\| \phi(z) \| < \delta_n$ *for* $|z| = 1$.

To see (11) note that if $y_j \in G_{n-1}$ there exists ϕ_j, a polynomial in $A_0(X)$ with $\phi_j(\bar{\Delta}) \subset \mathrm{co} = G_n$, $\phi_j(0) = y_j$ and $\| \phi_j(z) \| < \varepsilon$ for $|z| = 1$. If $y = c_1 y_1 + \cdots + c_N y_N$ then take $\phi = c_1\phi_1 + \cdots + c_N\phi_N$. If $|z| = 1$ then

$$\| \phi(z) \| \leqq N\varepsilon^{1/p} < \delta_n.$$

From (9), (10) we can repeat the original Roberts argument to show that if K_0 is the closure of $\bigcup_{n=0}^{\infty} \mathrm{co}\, G_n$ then K_0 is compact and convex and ext $K_0 \subset \{0\}$. If we set $K = \{\alpha u + \beta v \colon u, v \in K_0, |\alpha| + |\beta| \leqq 1\}$ then K is absolutely convex and compact and ext $K = \varnothing$.

Now suppose $h \colon K \to \mathbf{R}$ is a continuous plurisubharmonic function. Suppose y is in the absolutely convex hull of G_{n-1}. Then writing $y = \alpha y_1 + \beta y_2$ where $y_1, y_2 \in \mathrm{co}\, G_{n-1}$ and $|\alpha| + |\beta| \leqq 1$ we see from (11) that there is a polynomial $\phi \in A_0(X)$ with $\phi(\bar{\Delta}) \subset K$, $\phi(0) = y$ and $\| \phi(z) \| \leqq 2^{1/p}\delta_n$ for $|z| = 1$. The range of ϕ is contained in a finite-dimensional subspace E of the linear span of K. For $0 < \lambda < 1$, the range of $\lambda\phi$ is contained in the interior of $K \cap E$ relative to E. Thus, $h(\lambda\phi(z))$ is subharmonic on Δ and continuous on $\bar{\Delta}$. We conclude

$$\begin{aligned} h(\lambda y) &\leqq \max_{|z|=1} h(\lambda\phi(z)) \\ &\leqq \max_{\|x\|^p \leqq 2\delta_n^p} h(x). \end{aligned}$$

By continuity

$$h(y) \leqq \max_{\|x\|^p \leqq 2\delta_n^p} h(x).$$

Since $\mathrm{co}\, G_{n-1} \subset \mathrm{co}\, G_n \subset \cdots$ we conclude that $h(y) \leqq h(0)$. By density we conclude

$$h(0) = \max_{y \in K} h(y).$$

For any $y \in K$ the set $V = \{z : zy \in K\}$ is a convex neighborhood of zero in $\mathbf{C}$ and $x \to h(zy)$ is subharmonic on int V, continuous on V. Since it attains a maximum at 0, it is constant and so h is constant.

We conclude this section by quoting a result from [4].

PROPOSITION 3.3. *Let X be a separable p-normable quasi-Banach space. Then there is a transitive separable p-normable quasi-Banach space Y which contains a subspace linear isomorphic to X.*

REMARK. Y is called transitive if given any $y_1, y_2 \in Y$ with $y_1 \neq 0$ there is a linear operator $T: Y \to Y$ with $Ty_1 = y_2$. In fact Y can be chosen universal for all separable p-normable spaces.

Now the target is clear. If X contains just one non-zero analytic needle-point then Y will satisfy the conditions of Proposition 3.2.

4. The example

We first construct a functional version of the Ribe space ([11]; see also [9]). Since we are working over complex scalars we first note the inequality

$$|u \log|u| + v \log|v| + w \log|w|| \leqq \frac{2}{e}(|u| + |v|)$$

whereas $u, v, w \in \mathbf{C}$ with $u + v + w = 0$. Hence $0 \log 0$ is defined to be zero.

In fact we may suppose $|w| \geqq |u|, |v|$ and then note

$$\left| u \log \frac{|u|}{|w|} + v \log \frac{|v|}{|w|} \right| \leqq |u| \log \frac{|w|}{|u|} + |v| \log \frac{|w|}{|v|}$$

$$\leqq \frac{2}{e}|w| \leqq \frac{2}{e}(|u| + |v|),$$

since $x \log(1/x) \leqq (1/e)$ for $0 \leqq x \leqq 1$.

Let us define a functional $\Phi: L_2(\mathbf{T}) \to \mathbf{C}$ by

$$\Phi(f) = \int_{\mathbf{T}} f \log|f| \, dm - \left(\int_{\mathbf{T}} f \, dm \right) \log \left| \int_{\mathbf{T}} f \, dm \right|. \tag{12}$$

Now Φ is *quasilinear* for the L_1-norm (cf. [9], [5]), i.e.,

$$\Phi(\alpha f) = \alpha \Phi(f), \quad \alpha \in \mathbf{C}, \quad f \in L_2, \tag{13}$$

$$|\Phi(f_1 + f_2) - \Phi(f_1) - \Phi(f_2)| \leqq \frac{4}{e}(\|f_1\|_1 + \|f_2\|_1), \qquad f_1, f_2 \in L_2. \tag{14}$$

Thus we can form a twisted sum $\mathbf{C} \oplus_\Phi L_1(\mathbf{T})$ by completing $\mathbf{C} \oplus L_2(\mathbf{T})$ with respect to the quasi-norm

$$\|(\lambda, f)\| = |\lambda - \Phi(f)| + \|f\|_1. \tag{15}$$

The quasi-norm constant here is at most $4/e + 1$. We denote this constant by C.

Let us write RF for $\mathbf{C} \oplus_\Phi L_1$; we call this space the Ribe function space. The map $(\lambda, f) \to f$ extends to a quotient map Q_0: $\mathrm{RF} \to L_1$ with $\dim(\operatorname{Ker} Q_0) = 1$.

LEMMA 4.1. *Given $\varepsilon > 0$ there exist $0 < \beta < 1$ and $0 < \delta < 1$ such that if*

$$g(z) = \left(1, \beta\left(\sum_{n=1}^{\infty} \delta^n z^n \bar{w}^n + \sum_{n=1}^{\infty} \delta^n \bar{z}^n w^n\right)\right) \tag{16}$$

then g is continuous on $\bar{\Delta}$, harmonic on Δ, and

$$g(0) = (1, 0), \tag{17}$$

$$\|g(z)\| < \varepsilon, \qquad |z| = 1, \tag{18}$$

(19) *if $y \in \operatorname{co} g(\bar{\Delta})$ then there exists α, $0 \leqq \alpha \leqq 1$ with $\|y - \alpha(1, 0)\| < \varepsilon$.*

REMARK. This lemma implies $(1, 0)$ is a needle-point of RF, but it is not an analytic needle-point. In fact it may be shown that RF is A-convex.

PROOF. For any δ, $0 < \delta < 1$ set

$$1/\beta = \log \frac{1}{1 - \delta^2},$$

and define g by (16). We will show that for suitable δ, (17), (18), (19) hold. It is clear that g is harmonic on Δ and continuous on $\bar{\Delta}$, and that (17) holds.

Let $P(z, w)$ be the Poisson kernel, i.e.,

$$P(z, w) = \frac{1 - |z|^2}{(1 - \bar{w}z)(1 - w\bar{z})}, \qquad |w| = 1, \quad |z| < 1.$$

Let $P(z)$ be the corresponding function in $L_2(\mathbf{T})$. We note that since $\int P(z)dm = 1$

$$\Phi(P(z)) = \int_{\mathbf{T}} P(z) \log |P(z)| \, dm(w).$$

Now $\log|1 - \bar{w}z|$ and $\log|1 - w\bar{z}|$ are harmonic for $|w| < 1$ for fixed $z \in \Delta$. Hence $\log|P(z, w)|$ is also harmonic and

$$\int_{\mathbf{T}} P(z,w)\log|P(z,w)|\,dm(w) = \log|P(z,z)|$$
$$= \log\frac{1}{1-|z|^2}.$$

Now

$$g(z) = -(0,\beta) + (1, \beta P(\delta z)).$$

Hence if $|z| = 1$,

$$\| g(z) \| \leqq 2C\beta$$

where C is the quasi-norm constant.

Now suppose $y \in \operatorname{co} g(\bar{\Delta})$, say

$$y = \sum_{j=1}^{n} c_j g(z_j)$$

where $c_j \geqq 0$, $\Sigma\, c_j = 1$ and $z_j \in \bar{\Delta}$. Let

$$f = \sum_{j=1}^{n} c_j P(\delta z_j).$$

Then

$$y = -(0,\beta) + (1, \beta f).$$

The function $x \log x$ is convex for $x \geqq 0$. Hence as $f \geqq 0$ and $\int f = 1$,

$$\Phi(f) = \int f \log|f| \geqq \left(\int f\right) \log\left|\int f\right| = 0.$$

On the other hand

$$\Phi(f) \leqq \sum c_j \int P(\delta z_j)\log|P(\delta z_j)|\,dm$$
$$\leqq \sum c_j \log\frac{1}{1-\delta^2|z_j|^2}$$
$$\leqq \beta^{-1}.$$

Thus $0 \leqq \Phi(f) \leqq \beta^{-1}$. Let $\alpha = 1 - \beta\Phi(f)$. Thus

$$\| y - (\alpha, 0) \| < \| -(0,\beta) + \beta(\Phi(f), f) \|$$
$$\leqq 2C\beta.$$

Now for large enough δ (and small enough β), (18) and (19) hold.

The next theorem which seems to have some independent interest is the key to our construction.

THEOREM 4.2. *Let $f \in H_1$. Then*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \log |f(re^{i\theta})| \, d\theta = \sigma$$

exists and

$$|\sigma - f(0)\log|f(0)|| \leqq 2(\|f\|_{H_1} - |f(0)|).$$

REMARK. If we identify H_1 as a subspace of L_1, then this theorem implies $|\Phi(f)| \leqq 2 \|f\|_1$ for $f \in H_2$.

PROOF. The function $2|z| - z\log|z|$ is subharmonic on $\mathbf{C}$, as may be checked by differentiating and verifying the mean-value property at the origin. Thus if we set for $f \in H_\infty$

$$\Psi(f) = \int_{\mathbf{T}} 2|f| - (\operatorname{Re} f)\log|f| \, dm$$

then Ψ is plurisubharmonic on H_∞.

Now if $F: \Delta \to H_\infty$ is analytic then $\Psi \circ F$ is subharmonic. This is immediate for polynomials and follows in general by approximation. If $f \in H_1$ set

$$F(z)(w) = f(wz), \qquad w \in \mathbf{T}, \quad z \in \Delta.$$

Then

$$\Psi(F(z)) = \Psi(f_r)$$

where $r = |z|$ and $f_r(w) = f(rw)$. Hence $\Psi(f_r)$ is increasing in r. Hence it converges to a real limit or $+\infty$. The same argument can be applied to $(-f)$ and since $\int |f_r| \to \|f\|_{H_1}$ we conclude that

$$\lim_{r \to 1} \int (\operatorname{Re} f_r)\log|f_r| \, dm \quad \text{exists.}$$

Again arguing with if and $-if$ we finally deduce that

$$\lim_{r \to 1} \int f_r \log|f_r| \, dm = \sigma \quad \text{exists.}$$

Returning to $\Psi(f_r)$ we see that

$$2\|f\|_{H_1} - \operatorname{Re}\sigma \geqq 2|f(0)| - \operatorname{Re} f(0)\log|f(0)|$$

so that

$$\operatorname{Re}(\sigma - f(0)\log|f(0)|) \leqq 2(\|f\|_{H_1} - |f(0)|).$$

Applying the argument to λf where $|\lambda| = 1$ we deduce that

$$|\sigma - f(0)\log|f(0)|| \leqq 2(\|f\|_{H_1} - |f(0)|).$$

REMARK. This theorem may also be proved using Green's theorem, as was pointed out to the author by G. Weiss and M. Stoll.

We now return to the space RF. Let $M_0 \subset \mathbf{C} \oplus L_2$ by the subspace of all $(0, f)$ where $f \in H_2$. We claim that $(1, 0)$ is not in the closure M of M_0 in RF. In fact, if $f \in H_2$,

$$\begin{aligned} \|(1,f)\| &= |1 - \Phi(f)| + \|f\|_1 \\ &\geqq \max(1 - 2\|f\|_1, \|f\|_1) \\ &\geqq \tfrac{1}{3}. \end{aligned}$$

If $(0, f) \in M_0$

$$\begin{aligned} \|f\|_1 \leqq \|(0,f)\| &= |\Phi(f)| + \|f\|_1 \\ &\leqq 3\|f\|_1. \end{aligned}$$

Thus M is isomorphic to H_1 and Q_0 maps M isomorphically onto H_1.

Let Λ be the quotient space RF$/M$ and let Q_1: RF $\to \Lambda$ be the quotient map. Let $u = Q_1(1, 0)$, so that $u \neq 0$. Note that $\Lambda/[u] = L_1/H_1$.

PROPOSITION 4.3. *u is an analytic needle-point in Λ.*

PROOF. By Lemma 4.1, we can pick g to verify (16)–(19). Let $f = Q_1 g$. Then f is in $A_0(\Lambda)$ and satisfies (1)–(3). Thus u is an analytic needle-point.

THEOREM 4.4. *There is a twisted sum of $\mathbf{C}$ and L_1/H_1 which is not A-convex.*

THEOREM 4.5. *There exists a complex quasi-Banach space X and a non-empty compact absolutely convex subset K of X such that:*

(i) ext $K = \varnothing$.

(ii) *Every continuous plurisubharmonic function on K is constant.*

(iii) *K cannot be affinely embedded into L_0.*

PROOF. In view of the discussion in Section 3, (i) and (ii) are immediate. Let us suppose that there is a real-affine embedding $S: K \to L_{0,\mathbf{R}}$ into the space of real-measurable functions. We may suppose $S0 = 0$ so that S extends to a real-linear map $S_1: X_K \to L_{0,\mathbf{R}}$ where X_K is the linear span of K.

Now define $T: X_K \to L_{0,\mathbf{C}}$ (into the space of complex measurable functions) by

$$Tx = Sx - iS(ix);$$

T is complex-linear, and still an affine embedding of K into $L_{0,\mathbf{C}}$.

Consider X_K as a Banach space with the norm generated by K. By Nikishin's theorem ([10]) there exists $\phi \in L_{0,\mathbf{R}}$ with $\phi > 0$ a.e. so that $T_1 = \phi . T$ maps X_K boundedly into some $L_{p,\mathbf{C}}$ where $0 < p < 1$. Fix any q, $0 < q < p$. Then T_1 maps K homeomorphically into $L_{q,\mathbf{C}}$.

Now consider the map

$$\psi(x) = \| T_1 x \|_q^q.$$

ψ is plurisubharmonic on K and continuous. Further $\psi(0) = 0$. Hence $\psi \equiv 0$ and thus $T_1 \equiv 0$ which is a contradiction.

5. Concluding remarks

Theorem 4.2 seems to have some interesting ramifications. It is closely related to the work of Coifman–Rochberg [1] and Rochberg–Weiss [16], and has other applications to twisted sums. The author hopes to pursue these ideas in a future publication.

Probably the most obvious question which arises is whether the appearance of L_1/H_1 in Theorem 4.4 is a coincidence. The author suspects it is not.

CONJECTURE. *Every twisted sum of* $\mathbf{C}$ *and* L_1 *or* $\mathbf{C}$ *and* l_1 *is A-convex.*

The reason for this conjecture is that l_1 and L_1 are uniformly PL-convex (see [2]); it is known that every twisted sum of $\mathbf{C}$ and a uniformly convex Banach space is locally convex (see [5]). Thus the conjecture would follow by analogy.

REFERENCES

1. R. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.

2. W. J. Davis, D. J. H. Garling and N. Tomczak-Jagermann, *The complex convexity of quasi-normed linear spaces*, J. Functional Analysis **55** (1984), 110–150.

3. G. A. Edgar, *Complex martingale convergence*, in *Banach Spaces, Proc. Missouri Conference*, Springer Lecture Notes **1166**, Berlin, 1985, pp. 38–59.

4. N. J. Kalton, *Transivity and quotients of Orlicz spaces*, Comment. Math. (Special Issue in honor of W. Orlicz) (1978), 159–172.

5. N. J. Kalton, *The three space problem for locally bounded F-spaces*, Comput. Math. **37** (1978), 243–276.

6. N. J. Kalton, *An F-space with trivial dual where the Krein–Milman theorem holds*, Israel J. Math. **36** (1980), 41–50.

7. N. J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math. **84** (1986), 297–324.

8. N. J. Kalton and N. T. Peck, *A re-examination of the Roberts example of a compact convex set without extreme points*, Math. Ann. **253** (1981), 89–101.

9. N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Society Lecture Notes 89, Cambridge Univ. Press, 1985.

10. E. M. Nikishin, *Resonance theorems and superlinear operators*, Uspeki Mat. Nauk **25** (1970), 129–191 = Russian Math. Surveys **25** (1970), 124–187.

11. M. Ribe, *Examples for the nonlocally convex three space problem*, Proc. Amer. Math. Soc. **237** (1979), 351–355.

12. J. W. Roberts, *Pathological compact convex sets in the spaces* L_0, $0 < p < 1$, The Altgeld Book, University of Illinois, 1976.

13. J. W. Roberts, *A compact convex set with no extreme points*, Studia Math. **60** (1977), 255–266.

14. J. W. Roberts, *A nonlocally convex F-space with the Hahn–Banach approximation property*, in *Banach Spaces of Analytic Functions*, Springer Lecture Notes **604**, Berlin, 1977, pp. 76–81.

15. J. W. Roberts, *Cyclic inner functions in the Bergman spaces and weak outer functions in* H_p, $0 < p < 1$, Illinois J. Math. **29** (1985), 25–38.

16. R. Rochberg and G. Weiss, *Derivations of analytic families of Banach spaces*, Ann. Math. **118** (1983), 315–347.